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1987 J. Phys. A: Math. Gen. 20 3041

(http://iopscience.iop.org/0305-4470/20/10/046)

J. Phys. A: Math. Gen. 20 (1987) 3041-3045. Printed in the UK

COMMENT

Racah polynomials and a three-term recurrence relation for the Racah coefficients

K Srinivasa Rao†, T S Santhanam† and R A Gustafson‡

† Institute of Mathematical Sciences, Madras 600 113, India

‡ Department of Mathematics, Texas A & M University, College Station, TX 77843-3368, USA

Received 16 September 1986

Abstract. A three-term recurrence relation is derived for the Racah coefficients or 6-*j* symbols based on a set of orthogonal polynomials, called Racah polynomials, that generalise these coefficients. This relation is shown to be a consequence of the well known Biedenharn-Elliott identity.

The Racah coefficient or 6-*j* symbol has been shown (Srinivasa Rao *et al* 1975) to be expressible in terms of a set of three hypergeometric functions of unit argument as

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{E+1} M\Gamma(1-E)[\Gamma(1-A, 1-B, 1-C, 1-D, F, G)]^{-1} \times {}_4F_3(ABCD; EFG; 1) \qquad (1)$$

where

$$\begin{aligned} &M = \Delta(abc)\Delta(cde)\Delta(aef)(bdf) \\ &\Delta(xyz) = [(-x+y+z)!(x-y+z)!(x+y-z)!/(x+y+z+1)!]^{1/2} \\ &\Gamma(x, y, \ldots) = \Gamma(x)\Gamma(y)\ldots \\ &A = -R_{1k} \qquad B = -R_{2k} \qquad C = -R_{3k} \qquad D = -R_{4k} \\ &F = R_{kl} - R_{kk} + 1 \qquad G = R_{km} - R_{kk} + 1 \end{aligned} \qquad (2)$$

and E, determined from the Saalschutzian condition $A+B+C+D+1 = E+F+G$, as

$$E = -\sum_{j=1}^{4} R_{jk} + 2R_{kk} - R_{kl} - R_{km} - 1$$

$(klm) = (123)$ cyclically.

The R_{ik} in (2) are elements of the 4×3 array of Bargmann (1962) and Shelepin (1964):

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Vmatrix} d+f-b & e+f-a & e+d-c \\ a+f-e & b+f-d & a+f-c \\ d+c-e & b+c-a & b+d-f \\ a+c-b & e+c-d & a+e-f \end{Vmatrix} = \|R_{ik}\| \qquad (3)$$

with $i = 1, 2, 3, 4$ and $k = 1, 2, 3$.

0305-4470/87/103041+05\$02.50

The set of three ${}_4F_3(1)$ given in (1) has been shown (Srinivasa Rao *et al* 1975) to be necessary and sufficient to account for the known symmetries of the 6-j coefficient. It is guaranteed that, given a set of six angular momenta a, b, c, d, e, f, at least one of the three ${}_4F_3(1)$ belonging to this set is well defined. In this comment, we choose the set of numerator and denominator parameters (2) of the ${}_4F_3(1)$ corresponding to $(klm)=(312)$, without any loss of generality, to represent the 6-j coefficient.

To relate the 6-j symbol to the Racah polynomials, defined by Wilson (1980), we make use of the Bailey (1972) transform:

$$\begin{aligned}{}_4F_3(ABCD;\, EFG;\, 1)&\\ &=\Gamma(E+F-A-B-D,\, E+F-A-B-C,\, F-C-D,\, F)\\ &\quad\times[\Gamma(E+F-A-B,\, E+F-A-B-C-D,\, F-C,\, F-D)]^{-1}\\ &\quad\times{}_4F_3(E-B,\, E-A,\, C,\, D;\, E,\, E+F-A-B,\, E+G-A-B;\, 1)\end{aligned}\tag{4}$$

and replace c and f by $d+e-x$ and $b+d-n$, respectively, to obtain for (1)

$$\begin{aligned}\begin{Bmatrix} a & b & d+e-x\\ d & e & b+d-n\end{Bmatrix}&\\ &=(-1)^{a+b+e+d}M\Gamma(a+b+e+d+2,\, -2d+n,\, -2d+x,\\ &\quad N+1,\, N-n+x+1)[\Gamma(1+a+b-d-e+x,\, 1+x,\, 1+n,\\ &\quad 1-a-b-d+e+n,\, 1+N-n-x,\, 1+2d-n-x,\, -2d,\, 1+N-n,\\ &\quad -2d+x+n,\, 1+N-x)]^{-1}{}_4F_3(-2b-2d-1+n,\, -2d-2e-1+x,\\ &\quad -x,\, -n;\, -a-b-d-e-1,\, -2d,\, -2N;\, 1)\end{aligned}\tag{5}$$

where $N=b+e+d-a$ represents the number of terms,

$$0\leqslant x\leqslant N \qquad \text{and} \qquad 0\leqslant n\leqslant N.$$

The orthogonal polynomials defined by Wilson (1980) are

$$\begin{aligned}\mathcal{P}_n(t^2)=\mathcal{P}_n(t^2;\, a',\, b',\, c',\, d')&\\ &=\Gamma(a'+b'+n,\, a'+c'+n,\, a'+d'+n)[\Gamma(a'+b',\, a'+c',\, a'+d')]^{-1}\\ &\quad\times{}_4F_3(-n,\, a'+b'+c'+d'+n-1,\, a'-t,\, a'+t;\, a'+b',\, a'+c',\, a'+d';\, 1).\end{aligned}\tag{6}$$

In terms of this polynomial, the 6-j symbol (5) can be written as

$$\begin{aligned}\begin{Bmatrix} a & b & d+e-x\\ d & e & b+d-n\end{Bmatrix}&\\ &=(-1)^{a+b+d+e-n}\Delta(b,\, d,\, b+d-n)\Delta(a,\, e,\, b+d-n)\\ &\quad\times\Delta(a,\, b,\, d+e-x)\Delta(d,\, e,\, d+e-x)\Gamma(a+b+d+e-n+2)\\ &\quad\times[\Gamma(1+n,\, 1+n+a-b-d+e,\, 1+x,\, 1+2d-x,\, 1+N-x,\\ &\quad 1+x+a+b-d-e)]^{-1}\mathcal{P}_n(t^2;\, a',\, b',\, c',\, d')\end{aligned}\tag{7}$$

where

$$t=x-d-e-\tfrac{1}{2} \qquad a'=-d-e-\tfrac{1}{2} \qquad b'=-a-b-\tfrac{1}{2}$$

$$c'=a-b+\tfrac{1}{2} \qquad d'=-d+e+\tfrac{1}{2}.$$

Any set of orthogonal polynomials $\mathcal{P}_n(t^2)$ satisfy (see, for instance, Askey and Wilson 1979) a three-term recurrence relation:

$$t^2\mathcal{P}_n(t^2) = A_n\mathcal{P}_{n+1}(t^2) + B_n\mathcal{P}_n(t^2) + C_n\mathcal{P}_{n-1}(t^2) \tag{8}$$

for $n = 0, 1, \ldots$, with $\mathcal{P}_{-1}(t^2) = 1$. Wilson has shown that the polynomial $\mathcal{P}_n(t^2)$, defined in (6), satisfies the orthogonality property:

$$\frac{1}{2\pi\mathrm{i}}\int_C f(z)\mathcal{P}_m(z^2)\mathcal{P}_n(z^2)\,\mathrm{d}z = \delta_{mn}Rh_n \tag{9}$$

where

$$f(z) = \Gamma(a+z, a-z, b+z, b-z, c+z, c-z, d+z, d-z)[\Gamma(2z, -2z)]^{-1}$$

$$R = 2\Gamma(a+b, a+c, a+d, b+c, b+d, c+d)[\Gamma(a+b+c+d)]^{-1}$$

$$\begin{aligned} h_n = \Gamma(&n+1, a+b+c+d+2n-1, a+b+c+d+2n, a+b+n,\\ &a+c+n, a+d+n, b+c+n, b+d+n, c+d+n)\\ &\times[\Gamma(a+b, a+c, a+d, b+c, b+d, c+d, a+b+c+d)]^{-1} \end{aligned}$$

with a, b, c, d being complex and the contour defined as in Wilson (1980).

The coefficients A_n, B_n and C_n are then evaluated. A_n is determined by equating the highest power of t^2 to obtain

$$A_n = (a+b+c+d+n+1)[(a+b+c+d+2n)(a+b+c+d+2n-1)]^{-1}. \tag{10}$$

By repeated use of the orthogonality relation, in (8), we obtain

$$\begin{aligned} C_n &= h_nA_n/A_{n-1}\\ &= n(a+b+n-1)(a+c+n-1)(a+d+n-1)(b+c+n-1)(b+d+n-1)\\ &\quad\times(c+d+n-1)[(a+b+c+d+2n-1)(a+b+c+d+2n-2)]^{-1}. \end{aligned} \tag{11}$$

We note that when $t = a$, the polynomial (6) becomes

$$\mathcal{P}_n(a^2) = \Gamma(a+b+n, a+c+n, a+d+n)[\Gamma(a+b, a+c, a+d)]^{-1}$$

so that evaluation of the three-term recurrence relation (6), at $t = a$, yields for B_n

$$\begin{aligned} B_n = a^2 - &(a+b+n)(a+c+n)(a+d+n)(a+b+c+d+n-1)\\ &\times[(a+b+c+d+2n)(a+b+c+d+2n-1)]^{-1}\\ &-n(b+c+n-1)(b+d+n-1)(c+d+n-1)\\ &\times[(a+b+c+d+2n-1)(a+b+c+d+2n-2)]^{-1}. \end{aligned} \tag{12}$$

Having determined A_n, B_n and C_n, we now use the three-term recurrence relation for $\mathcal{P}_n(t^2)$ in conjunction with (7) which expresses $\mathcal{P}_n(t^2)$ in terms of the 6-j symbol. After a straightforward calculation and simplifications, we obtain, on resubstituting $x = d+e-c$ and $n = b+d-f$, the new recurrence relation satisfied by the 6-j symbol as

$$\begin{aligned}&[2f(f+1)(2f+1)(c-d-e)(c+d+e+1)\\&\quad+(f+1)(-a+e+f)(a+e+f+1)(-b+d+f)(b+d+f+1)\\&\quad+f(b+d-f)(b-d+f+1)(a+e-f)(a-e+f+1)]\begin{Bmatrix}a&b&c\\d&e&f\end{Bmatrix}\\&=(f+1)\square(a,e,f)\square(b,d,f)\begin{Bmatrix}a&b&c\\d&e&f-1\end{Bmatrix}\\&\quad+f\square(a,e,f+1)\square(b,d,f+1)\begin{Bmatrix}a&b&c\\d&e&f+1\end{Bmatrix}\end{aligned}\tag{13}$$

where we have introduced the notation

$$\square(x,y,z)=[(-x+y+z)(x-y+z)(x+y-z+1)(x+y+z+1)]^{1/2}.$$

The relation (13), which obviously holds only for $f\geqslant 1$, is a three-term recurrence relation in f. In principle, this recurrence relation can be used to extend the tables of 6-j symbols. Raynal (1979) has obtained simple recurrence relations, valid for any arguments in terms of Whipple's parameters, though not a recurrence relation in which only one argument changes as $f-1$, f and $f+1$.

We now show that (13) can also be shown to be a consequence of the Biedenharn (1953)-Elliott (1953) identity for the Racah coefficient:

$$\begin{aligned}&W(a'ab'b;c'e)W(a'ed'd;b'c)\\&\quad=\sum_g(2g+1)W(abcd;eg)W(c'bd'd;b'g)W(a'ad'g;c'e)\end{aligned}\tag{14}$$

which 'is the key relationship for elevating the study of Racah coefficients to a position that is independent of the concept of Wigner coefficient' (Biedenharn and Louck 1981). In (14), we set $a'=a$, $b'=b$, $d'=f$ and $c'=1$ to obtain

$$\begin{aligned}&W(aabb;1e)W(aefd;bc)\\&\quad=\sum_g(2g+1)W(abcd:eg)W(1bfd;bg)W(aafg;1c).\end{aligned}\tag{15}$$

We now substitute the special values of the Racah coefficient on the right- and left-hand sides of the above identity having one of the arguments equal to 1, given in the table of Biedenharn *et al* (1952).

After algebraic simplification we obtain

$$\begin{aligned}&(2f+1)\{[b(b+1)-d(d+1)+f(f+1)][a(a+1)-c(c+1)+f(f+1)]\\&\quad-2f(f+1)[a(a+1)+b(b+1)-e(e+1)]\}W(abcd;ef)\\&=(f+1)\square(b,d,f)\square(a,c,f)W(abcd;ef-1)\\&\quad+f\square(b,d,f+1)\square(a,c,f+1)W(abcd;ef+1).\end{aligned}\tag{16}$$

A comparison of (16) with (13) shows that, since the right-hand sides of the two expressions are identical provided, in (16), c and e are interchanged and the relationship of the Racah coefficient to the 6-j coefficient is used:

$$W(abcd;ef)=(-1)^{a+b+c+d}\begin{Bmatrix}a&b&c\\d&e&f\end{Bmatrix}$$

we must have

$$\begin{aligned}(2f+1)\{[b(b+1)-d(d+1)+f(f+1)][a(a+1)-e(e+1)+f(f+1)]\\ -2f(f+1)[a(a+1)+b(b+1)-c(c+1)]\}\\ \equiv 2f(f+1)(2f+1)(c-d-e)(c+d+e+1)\\ +(f+1)(-a+e+f)(a+e+f+1)(-b+d+f)(b+d+f+1)\\ +f(a+e-f)(a-e+f+1)(b+d-f)(b-d+f+1).\end{aligned} \tag{17}$$

That this identity holds can be seen when the left- and right-hand side expressions are both expanded as polynomials in f. This establishes the validity of (13).

Our derivation of the recurrence relation (13) is direct from the generalised (Racah) orthogonal polynomial, which satisfies the three-term recurrence relation (8).

The authors are grateful to the referee for valuable comments.

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